hep-th/0006224

IASSNS-HEP-00/50, RUNHETC-2000-27

D-branes on Stringy Calabi–Yau Manifolds

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We argue that D-branes corresponding to rational B boundary states in a Gepner model can be understood as fractional branes in the Landau–Ginzburg orbifold phase of the linear sigma model description. Combining this idea with the generalized McKay correspondence allows us to identify these states with coherent sheaves, and to calculate their K-theory classes in the large volume limit, without needing to invoke mirror symmetry. We check this identification against the mirror symmetry results for the example of the Calabi–Yau hypersurface in $\mathbf{W}\mathbb{P}^{1,1,2,2,2}$.

June 2000

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1. Introduction

D-branes in Calabi–Yau compactification of string theory have been the focus of a number of recent works. In this work we continue the study of D-branes at Gepner points initiated in [33,4]. We will show how many results for the spectrum of rational boundary states and the corresponding brane world-volume theories can be derived starting from the linear sigma model. The basic idea is to realize the boundary states as fractional branes in the Landau–Ginzburg orbifold phase; we will show how recent mathematical work on the generalized McKay correspondence determines the identification of these boundary states as bundles in the large volume limit, and check this identification in an example against results obtained using mirror symmetry. As in [15], this framework allows identifying bound states of branes with bundles and provides explicit descriptions of their moduli spaces; we will pursue this in more detail in subsequent work.

For an overview of this line of work, we refer to [12]. The starting point is Gepner's identification of certain $\mathcal{N} = 2$ CFT's as stringy Calabi–Yau manifolds (CYs). In [33], rational boundary states (those which can be easily obtained as orbifold products of boundary states in the individual $\mathcal{N} = 2$ minimal models) were constructed for Gepner models. This provides explicit CFT realizations of D-branes on these manifolds, and allows computing RR charges (in a natural basis at the Gepner point), as well as the number of marginal operators. It is also possible to compute superpotentials, as outlined in [4] and as has been done in examples [5].

The natural extension of Gepner's identification would be to identify these BPS boundary states with specific D-branes in the large volume limit of the same Calabi–Yau. The work [4] made first steps towards such an identification. The "decoupling conjecture" made there gives strong reasons to think that B branes at any point in Kähler moduli space should be identifiable with specific holomorphic objects (bundles, coherent sheaves or complexes) in the large volume limit. Using a derivation of the Kähler moduli space from mirror symmetry [7], an explicit translation of the RR charges of the B boundary states in the $(3)^5$ Gepner model into Chern classes was made, which determines the topological type of the corresponding bundles in the large volume limit. Similar results for other Calabi–Yau manifolds have been obtained in [10,25,38,32].

In [15] the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold was studied in detail, and a remarkable relation was found between the quiver gauge theory of [36,37,16] and Beilinson's construction [2] of holomorphic vector bundles on $\mathbb{P}^2$: the quiver theory and mirror symmetry results reproduce this construction, providing a very detailed correspondence between F-flat configurations of the gauge theory and holomorphic bundles in the large volume limit. It was also pointed out that the results of [4] for the quintic had a very similar relationship to Beilinson's construction of bundles on $\mathbb{P}^4$.

The present work will explain and generalize this relationship. Besides the work above, it is inspired by a generalization of Beilinson's construction developed in recent mathematical work on the generalized McKay correspondence [34,23,3].

The basic idea is to realize the Calabi–Yau threefold of interest as a submanifold of the resolution of a higher dimensional orbifold $\mathbb{C}^n/\Gamma$, define D-branes in the higher dimensional orbifold using the construction of Douglas and Moore [17], and then identify the D-branes of interest as the restriction of these to the original CY. As we explain, this procedure can also be directly motivated by the physics of boundary states in the linear sigma model construction of the CY [40,20,19,21].

Acknowledgements. We would like to express our special thanks to Dave Morrison for collaboration on the early stages of this project, and for many valuable discussions and suggestions.

2. Gepner models and quivers

2.1. Gepner models and linear sigma models

A Gepner model is a product of r minimal models at level k_i whose central charges $3k_i/(k_i+2)$ add to $3n$. As we will review shortly, this corresponds to a Fermat hypersurface in a weighted projective space, which if $n+r$ is even is $\mathbf{W}\mathbb{P}(w_i)$, where $w_i = K/(k_i+2)$ and $K = \mathrm{lcm}\{k_i+2\}$. If $n+r$ is odd, we adjoin $w_{r+1} = K/2$ to this list, and henceforth take $n+r$ even. One can show that for $r = n+2$, these requirements imply that $K = \sum w_i$.

When $n = 3$, such a Gepner model can also be realized as a $(2,2)$ linear sigma model [40]. It has a $U(1)$ gauge group, r chiral superfields Z^i with charges w_i, and a chiral superfield P with charge $-K$. There is a superpotential $W_G = P\sum_i (Z^i)^{k_i+2}$. The D-flatness conditions are

$$\zeta = \sum_i w_i |Z^i|^2 - K|P|^2 \tag{2.1}$$

with an FI parameter ζ, and the model has two phases depending on this parameter. The Gepner model is associated with the "Landau–Ginzburg" phase with $\zeta < 0$ and

$\langle P \rangle \neq 0$; the action expanded around this configuration is a sum of $\mathcal{N} = 2$ Landau–Ginzburg models, while the $U(1)$ symmetry is broken to $\mathbb{Z}_K$. On the other hand, $\zeta > 0$ produces the "geometric" phase in which D-flat configurations with $P = 0$ parameterize the weighted projective space $\mathbf{W}\mathbb{P}(w_i)$, while the condition $0 = \partial W_G / \partial P$ defines the CY as a hypersurface in this space.

It will be useful to have a picture of the space of D-flat configurations (in other words, the vacua of the corresponding theory with no superpotential) in the two phases. The general D-flat configuration in the geometric phase allows $P \neq 0$ and the total space is a line bundle over $\mathbf{W}\mathbb{P}(w_i)$. For $K = \sum w_i$ this is the anticanonical bundle, and the total space is itself a Calabi–Yau, generically singular because of the singularities of $\mathbf{W}\mathbb{P}(w_i)$. Let us denote this Calabi–Yau as $X(w_i)$ or simply X.

Similarly, in the Landau–Ginzburg phase the general D-flat configuration has $Z^i \neq 0$. The condition (2.1) determines P and since $\langle P \rangle \neq 0$ always, the $U(1)$ gauge symmetry is always broken to $\mathbb{Z}_K$. Thus the D-flat moduli space in this phase is $\mathbb{C}^r / \mathbb{Z}_K$ with the $\mathbb{Z}_K$ action defined by the action of a generator

$$g(Z^i) = e^{2\pi i w_i / K} Z^i.$$

Note that this generates a discrete subgroup of $SU(r)$, so this noncompact orbifold is also a CY.

In a later section, we will review the toric description of these configuration spaces and the relation between these two phases. The general idea is that the algebra of holomorphic functions on the configuration space is independent of the D-flatness conditions, and thus must be the same in the two phases. Thus we can consider the space $X(w_i)$ as a (partial) resolution of the noncompact orbifold $\mathbb{C}^r / \mathbb{Z}_K$. In both phases, the superpotential will confine the theory to the CY_3 as a hypersurface in the exceptional divisor, $Z^i = 0$ in $\mathbb{C}^r / \mathbb{Z}_K$, and the resolution of this point $\pi^{-1}(0)$ in X.

2.2. B boundary states

Our basic claim is that the rational B boundary states can be thought of as the restriction of the "fractional brane" states of the $\mathbb{C}^r / \mathbb{Z}_K$ orbifold to the CY_3.

A fractional brane state in a $\mathbb{C}^r / \Gamma$ orbifold is a Dirichlet boundary state in $\mathbb{C}^r$, with an additional choice of an irreducible representation of Γ. A collection of fractional branes is labeled by a representation R of Γ or equivalently the multiplicities n_a of the irreps γ_a.

The world-volume theory of the collection is then derived from the world-volume theory of $\dim R$ branes in $\mathbb{C}^r$ by projecting on invariants under the action of Γ on the fields twisted by the γ_a's; in particular vectors Z^i in $\mathbb{C}^r$ (such as those parameterizing transverse motion of the branes) are projected as

$$\gamma_R(g)^{-1} Z^i \gamma_R(g) = (\gamma_{def})^i_j(g) Z^j.$$

These definitions do not require $\hat{c} \leq 10$ or even that the bulk theory of interest be a conformal field theory. Thus we can apply them directly to the LG orbifold phase of the linear sigma model. It is known that Dirichlet boundary conditions are $\mathcal{N} = 2$ supersymmetric in the ungauged LG model [39,20,21]. If we work far below the scale of $U(1)$ gauge symmetry breaking (set by the vev of P and thus the FI term), the full (B type) linear sigma model boundary conditions must reduce to conventional Dirichlet boundary conditions for Z^i. We need only take the unbroken discrete gauge symmetry into account, which is what is done by the fractional brane prescription.

Now, since we start with a non conformal theory, we must expect the IR spectrum of marginal operators to be rather different from the UV free theory spectrum, raising the question of what world-volume theory we should take for the branes.

We do know that the flow must preserve the massless Ramond states, as these are protected by the usual index considerations. We can thus compute the massless Ramond spectrum in the UV and carry it to the IR.

We then make the crucial assumption that—although the combinations of BPS branes we are considering together break all supersymmetry (they preserve different $\mathcal{N} = 1$ subalgebras of the original $\mathcal{N} = 2$)—this supersymmetry breaking is a spontaneous supersymmetry breaking in an effective $\mathcal{N} = 1$, $d = 4$ world-volume theory. In particular, combinations of BPS branes which together would break supersymmetry can lead to BPS bound states, which are simply described by (quasi)-supersymmetric vacua of the combined theory. This assumption is not completely obvious, especially as we will be discussing fields with string scale masses in the broken supersymmetry vacua, and as we will see it is literally true only for a subset of the theories. It is further discussed and motivated in [13]. In any case, we proceed to postulate an effective $\mathcal{N} = 1$ world-volume theory which is compatible with our information.

The massless open string Ramond sector for the CFT of r free superfields will simply be a spinor (of definite chirality) of $SO(2r)$, or equivalently a sum of antisymmetric representations of $SU(r)$. In the familiar case of $\mathbb{C}^3$ orbifolds, this leads to a singlet and a vector of $SU(3)$, and supersymmetry incorporates these into space-time vector and chiral multiplets respectively. The resulting world-volume theory is the familiar $\mathcal{N}=4$ super Yang–Mills and its dimensional reductions, to which the Γ projection is applied.

In the case of $\mathbb{C}^5$ orbifolds, these considerations lead to a vector of $SU(5)$, a three index antisymmetric tensor, and a singlet (equivalently a five index antisymmetric tensor). We then assume that the flow to the IR leads to a $(2,2)$ supersymmetric theory with an $\mathcal{N}=1$, $d=4$ world-volume interpretation. On general grounds, the nonsinglets will have to enter into chiral multiplets in this theory. The singlet might enter into either chiral or vector multiplets *a priori*, but given that any boundary theory will contain the operator 1 which is the internal CFT part of the gauge boson vertex operator, there must be a vector multiplet in the space-time theory, whose fermion must be this singlet.

This motivates the claim that the world-volume theory of N D-branes on $\mathbb{Z}^5/\Gamma$ (assuming $\mathcal{N}=1$ supersymmetry) is a $U(N)$ gauge theory with 15 chiral multiplets in the adjoint of $U(N)$, transforming as $\mathbf{5}+\overline{\mathbf{10}}$ of a global $SU(5)$. Let us denote these multiplets as X^i and $Y^{[ij]}$ respectively.

Such a theory admits gauge invariant superpotentials, and the leading possible term is cubic:

$$W = \mathrm{tr}\ X^i X^j Y^{[ij]} + \ldots. \tag{2.2}$$

Such a term in the superpotential is also natural from the CFT point of view and as discussed in [4], it can be computed in the topologically twisted model. The non-zero amplitudes are those in which the operators combine to saturate the fermion zero modes; in terms of the translation to forms on $\mathbb{C}^5$ given above they are the amplitudes in which the product of the forms involved produces a top form on $\mathbb{C}^5$, which produce exactly (2.2).

2.3. Quiver gauge theory

We now apply the orbifold projection to derive the world-volume theory of boundary states on $\mathbb{C}^5/\Gamma$. We will discuss $\Gamma \cong \mathbb{Z}_K$ in detail here, though similar considerations would apply to nonabelian groups, as for $\mathbb{C}^2/\Gamma$ in [24]. It will be a quiver theory with K nodes, chiral superfields in the $\mathbf{5}$ of $SU(5)$ $X^i_{M,M+w_i}$, chiral superfields in the $\overline{\mathbf{10}}$ $Y^{ij}_{M,M-w_i-w_j} \equiv -Y^{ji}_{M,M-w_i-w_j}$, and the restriction of (2.2),

$$W = \sum_{M,i,j} X^i_{M,M+w_i} X^j_{M+w_i,M+w_i+w_j} Y^{ij}_{M+w_i+w_j,M}. \tag{2.3}$$

Indeed all of this data agrees with the explicit CFT results of [33,4]. B boundary states in these models are characterized by labels L_i in each minimal model factor and $M = \sum w_j M_j$ in $[0, 2K-1]$. In particular, if we define the states $|M\rangle$ with a given M and all $L_i = 0$, and the operator g acting as $M \to M+2$, then we can write the intersection matrix between the $L=0$ states as

$$\begin{aligned} I_G &= \prod_j (1 - g^{w_j}) \\ &= \sum_{k_j=0,1} (-)^{\sum k_j} g^{\sum k_j w_j} \end{aligned} \tag{2.4}$$

which agrees with the massless Ramond spectrum we described. This term in the superpotential can be checked from CFT [5].

In general the superpotential will contain higher order terms as well. These are computable in CFT and are also topological, but not too much is known about them at present. Our results so far are consistent with the idea that in the theories in which the low energy description is justified (we will explain this point shortly), such terms are absent, but this remains to be seen.

The final item required to complete the specification of the world-volume gauge theories is the Fayet-Iliopoulos terms for the $U(1)^K$ subgroup of the gauge group. As pointed out in [14,13], these can be determined from the masses of the bosonic superpartners of the massless fermions, which are also known from CFT. These superpartners can be defined using the spectral flow operator on either brane (they produce the same result up to a phase) and in the Gepner models under discussion are obtained by multiplying the top chiral primaries in a subset of the individual minimal model factors. The result is that a boson on a link from M to $M+w$ has $m^2 = -w\lambda$ where we consider $X^i_{M,M+w_i}$ as "forward" links (so these are tachyonic) and $Y^{[ij]}_{M,M-w_i-w_j}$ as "backward" links (so these are massive). λ is computable and order string scale.

The FI terms must then reproduce these masses

$$m^2_{ij} = \zeta_i - \zeta_j$$

where ζ_i is the FI term for the $U(1)$ of the i'th brane. In fact one can argue directly for this structure from general properties of $\mathcal{N}=2$ CFT: the ζ_i and thus m^2 are directly related to the overall $U(1)$ charge and thus the phases of the central charges of the two branes [13].

This determines the FI terms to be $\zeta_k = k\lambda$, but one immediately notices that this cannot reproduces all the masses in all theories. The condition for it to work is that we do not have links going "all the away around the clock," for example closed loops with only X fields. It does not forbid closed loops involving both X and Y.

This is a condition on the allowed fractional brane content: if all types of fractional branes are present it will fail, but it can be satisfied by excluding some types of fractional branes. If it fails, the masses cannot all be reproduced in this low energy field theory description, which probably signals its breakdown. As we will see, such configurations typically restrict to branes on the Calabi-Yau with zero RR charge (or with simpler realizations) and would thus be expected to decay to the vacuum (or the simpler realization); this process would then not have a description purely in the low energy theory.

In conclusion, we have derived the low energy theories of combinations of the $L = 0$ Gepner model branes by adapting the orbifold construction to $\mathbb{C}^5/\mathbb{Z}_K$ in a way suggested by linear sigma model considerations. The result is a quiver gauge theory very analogous to those for $\mathbb{C}^3/\Gamma$ orbifolds. The construction generalizes straightforwardly to general quotients $\mathbb{C}^r/\Gamma$.

2.4. Bound states

We now make the general claim that supersymmetric vacua of these theories with unbroken gauge symmetry $U(1)$ correspond to general (classical) bound states of these rational branes. If one keeps all the modes of the open string theory (e.g. by using string field theory), this claim seems difficult to dispute. A less obvious claim is that many bound states can be described purely within the theory obtained by keeping the chiral primaries, in other words the theories we just derived. The potential problem is that the FI terms and thus the vevs of the fields at the supersymmetric vacuum have string-scale values.

Nevertheless, as we discussed, in a large subset of the theories we discussed, those with chiral fields whose masses can be reproduced by FI terms, there is no *a priori* reason for this description to break down. A basic test of it which can be done is to construct non-rigid ($L > 0$) rational branes as bound states of the $L = 0$ branes and check that the dimension of the moduli space comes out right. Some non-trivial examples of this for the quintic were discussed in [15], and further examples will be discussed in [9].

Now, work on non-BPS brane configurations in flat space and other simpler examples does support the claim that in many cases a good qualitative description can be obtained just using tachyons and massless fields. Thus it should probably not be too surprising that this works in some subset of the theories we just derived. Whether this works in all the theories which *a priori* appear sensible, and whether string field theory or similar frameworks can provide a more general description, are important questions for further work.

3. The large volume interpretation of the fractional branes

In [15], it was noticed that the large volume interpretation of the fractional branes of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold found using mirror symmetry [10] gave the same bundles which form the natural basis (due to Beilinson) for the general construction of bundles on $\mathbb{P}^2$, which is the exceptional divisor of the resolution of $\mathbb{C}^3/\mathbb{Z}_3$.

As it turns out, recent mathematical work has led to a very general conjecture for the higher dimensional analog of this correspondence, which we will be able to apply directly to our $\mathbb{C}^5/\Gamma$ theories [34,3,23].

The idea is to generalize the famous McKay correspondence [29] between discrete subgroups Γ of $SU(2)$ and finite simply laced Dynkin diagrams to discrete subgroups $\Gamma \subset SU(N)$. More specifically, one has a relation between the representation ring of Γ (which is directly encoded in the Dynkin diagram) and a basis for the exceptional cycles in a resolution of a $\mathbb{C}^2/\Gamma$ singularity. According to [34], the idea that a higher dimensional generalization should exist actually has its origins in the study of orbifolds in string theory, and the observation that the Euler number of a resolved $\mathbb{C}^3/\Gamma$ singularity equals the number of conjugacy classes of Γ.

The precise version of this idea which has been generalized is a duality between the category of sheaves on $X \cong \mathbb{C}^n/\Gamma$, and the category of sheaves with compact support, which for $X \cong \mathbb{C}^n/\Gamma$ will be sheaves supported at the origin (or, if a partial resolution has been performed, on the exceptional divisor).

In making this precise, one must work with specific categories. The duality can be made quite concrete, as was done by Ito and Nakajima in [23], where the dual objects are constructed as explicit complexes of line bundles. This should allow making a detailed identification between quiver representations and large volume sheaves along the lines of [15]. In [3], the duality was shown to be an equivalence between derived categories, which should allow proving the analog of Beilinson's theorem for this case. Although less concrete, this is still a very strong statement about the relation between the two categories.

Here we will content ourselves with deriving the K-theory classes which correspond to the dual basis. We will then restrict these to the Calabi–Yau and compare with the predictions of mirror symmetry in a solved example.

The mathematics of the generalized McKay correspondence is clearly described in [34,3,23] and thus in the remainder of this section we concentrate on describing the ideas for physicists.

3.1. Orbifold resolution, tautological line bundles, and dual bases

The problem of resolving singularities $X = \mathbb{C}^2/\Gamma$ has a long mathematical history, going back to Klein (see [35] for some of this background). Such a singular variety X can be resolved to a smooth space Y with non-trivial $H^2(Y)$ and intersection form given by the Cartan matrix of the extended ADE Dynkin diagram associated to the subgroup $\Gamma \in SU(2)$.

The most basic string theory application of this is to the duality between IIa strings on K3 and heterotic strings on T^4 [22], where IIa D2-branes wrapped on the resolved (or "exceptional") cycles provide the nonabelian gauge bosons of the enhanced ADE gauge symmetry predicted by duality.

The most direct connection between this geometry and the structure of the group Γ appears in the McKay correspondence. The McKay quiver associated to Γ has a node for every irrep r_i of Γ, and a link from r_i to r_j for every component r_j in $r_{def} \otimes r_i$, where r_{def} is the representation by which Γ acts on $\mathbb{C}^2$. The result is a quiver which can be simply obtained from the ADE extended Dynkin diagram by replacing each link of the latter by a pair of links of opposite orientation.

This construction can also form the basis of an explicit construction of the resolved space, as was done by Kronheimer [26,27]. Physically the same construction appears in defining D-branes on the quotient space, and provides an explicit gauge theory description of the resolution [17]. It furthermore provides an explicit description of the branes wrapped on the exceptional cycles; these are "fractional branes" obtained by using irreducible representations in the quotient construction. One thus has the basic prediction that there should exist a natural basis for $H^2(Y)$, or better the K-theory of the resolved singularity, labeled by irreducible representations of Γ.

In this context the relation between the Dynkin diagram and the intersection form has a physical interpretation as well: each link corresponds to a hypermultiplet coming from open strings stretched between a pair of fractional branes; their number can be computed from the index theorem and is equal to the intersection number.

An even more general physical system was studied in [17], containing both Dp-branes at points in X and D$p+4$-branes extending in X. It was found to reproduce a construction of general self-dual gauge fields on the resolved singularity due to Kronheimer and Nakajima [28]. Now both types of branes are labeled by a choice of group representation, and each can be associated to a quiver node. Let R_i be the D$p+4$ node corresponding to r_i and S^j be the Dp node corresponding to r_j. The spectrum of $(p, p+4)$-strings between a pair (R_i, S^j) is also determined by the orbifold projection and one finds the number of hypermultiplets to be δ_i^j. As in our previous discussion, this implies that the intersection form between the two types of branes should be

$$\langle R_i, S^j \rangle = \delta_i^j. \tag{3.1}$$

Now the interpretation of the D$p+4$ (extended) branes as bundles is rather clear, at least far from the singularity. The orbifold projection acts on the Yang–Mills connection as

$$\gamma A_i(z) \gamma^{-1} = r_i^j A_j(g(z)). \tag{3.2}$$

This tells us that scalar matter in the fundamental, i.e., a section of the associated bundle, must transform as

$$\gamma \phi(z) = \phi(g(z)). \tag{3.3}$$

A particularly simple case is to take γ to be the regular representation, in which case we can consider $\phi(z)$ as a vector-valued field indexed by an element of Γ, so (3.3) becomes

$$\phi_{gh}(z) = \phi_h(g(z)). \tag{3.4}$$

This bundle is referred to as the "tautological bundle" over the quotient space. It can be decomposed as a direct sum over bundles R_i associated to irreps γ which if Γ is abelian are line bundles; these are the tautological line bundles.

The dual relation (3.1) then determines the bundles S_j. On a noncompact space X, the natural duality for K-theory (just as for cohomology) is between $K(X)$ and the K-theory of bundles with compact support $K_c(X)$, meaning bundles over compact submanifolds of X. Thus the bundles S^j naturally live in $K_c(X)$ and provide a preferred basis for it. These are the bundles associated to the fractional Dp-branes.

Given the intersection form in an explicit basis, we can make this definition quite concrete. For example, if we have

$$\langle R_i, R_j \rangle \equiv (I^{-1})_{ij}, \tag{3.5}$$

then we can write

$$S^j = I^{ij} R_i \tag{3.6}$$

for which

$$\langle S^j, S^k \rangle = I^{jk}. \tag{3.7}$$

As in [23], the relation (3.6) can be used to define the S^j as complexes built from the bundles R_i. In terms of the K-theory classes, (3.6) becomes

$$[S^j] = I^{ij} [R_i], \tag{3.8}$$

a simple explicit formula for the K-theory classes of the fractional branes given those of the tautological line bundles.

Restricting these bundles or their classes to a subvariety, such as a Calabi–Yau embedded in the exceptional divisor, is a standard operation: let $V^j = S^j|_{CY}$ be these restrictions. Thus we can define an intersection form on the Calabi–Yau

$$I_{CY}^{jk} = \langle V^j, V^k \rangle_{CY} = \int_{CY} \mathrm{ch}(\overline{V}^j)\mathrm{ch}(V^k)\mathrm{Td}(CY), \tag{3.9}$$

and the conjecture is that

$$I_{CY} = I_G \tag{3.10}$$

where I_G is the intersection form (2.4) of section 2.

The result is a physically motivated prediction for the K-theory classes of the rational B boundary states, which we will test against results derived using mirror symmetry. Indeed, the example of the quintic discussed in [15] is already a non-trivial test, as the procedure we just described leads to Beilinson's dual bases in the case of $\mathbb{C}^n/\mathbb{Z}_n$, which as checked there agree with the results of [4].

Although (3.8) is the formula we will test in this paper, let us emphasize that (3.6) provides a definition of the fractional branes S^j as holomorphic objects, not just K-theory classes. This is made quite explicit in [3,23], where the dual bases in (3.1) are used to construct a resolution of the diagonal, which can be used to prove Beilinson's theorem for these spaces. This leads to explicit large volume interpretations of general bound states of the fractional branes as complexes of sheaves, as we will discuss in future work [9].

So far, none of our definitions had any real dependence on the dimension of X; we could make the same discussion for $\mathbb{C}^n/\Gamma$ for any n. The point where such dependence will come in is when we discuss the resolution of the singular space X in detail. Indeed, unless we can resolve X, it is not obvious in what sense the R_i can be thought of as bundles or how to compute their K-theory classes. General theory [18] does tell us that given a resolution Y of X, there will be a natural lift of these K-theory classes to Y, but we might expect this to depend on the particular resolution we choose.

Thus we need to discuss the resolution of X in more detail. One idea which has been used with great success in the math literature has been to use subspaces of the Hilbert scheme of $N = |\Gamma|$ points on $\mathbb{C}^n$ which are invariant under Γ. It has been shown for $n = 2$ and all Γ, and $n = 3$ and abelian Γ, that such a subspace provides a canonical complete resolution Y of X. The definition of tautological bundle then lifts naturally to Y, and the story can be completed in this framework.

For $n > 3$ there are known examples in which this construction does not produce a complete resolution. Moreover, the Hilbert scheme becomes progressively more difficult to work with in higher dimensions.

An alternate approach is to define the quotient X as the moduli space of a quiver gauge theory, and then find the resolution Y by the usual procedure of turning on Fayet–Iliopoulos terms. This approach was successfully used for $\mathbb{C}^3/\Gamma$ by Ito and Nakajima and is clearly well motivated in our D-brane application, so we shall follow it below. One disadvantage of this approach is that the choice of Fayet–Iliopoulos terms generally translates into a choice of resolution; it is not obvious that any of these is preferred. However, as we argued, the Gepner models produce quiver gauge theories come with a natural choice of FI terms, so we should try to make the construction work with these.

4. Orbifolds via toric methods

A general procedure for analyzing abelian orbifolds as toric varieties was given in [16]. We will briefly review this, and give the definition of the tautological line bundles in this context.

D-branes in orbifold backgrounds are described by supersymmetric world-volume gauge theories, as was shown in [17] for orbifolds in flat space and as we have argued here for Landau–Ginzburg orbifolds. The resolved orbifold will be the moduli space of supersymmetric vacua of the regular representation theory.

In physical terms, a toric variety can be defined as the moduli space of vacua for an abelian $\mathcal{N}=1$ supersymmetric gauge theory with no superpotential. While the moduli space of vacua for a general supersymmetric gauge theory does not admit a toric realization, theories for which the F-flatness constraints can be written as relations between monomials do.

In the general class of theories we described, the F-flatness conditions are indeed relations between monomials: they are

$$X^i_{M,M+w_i} X^j_{M+w_i,M+w_i+w_j} = X^j_{M,M+w_j} X^i_{M+w_j+w_i}, \tag{4.1}$$

where $M = 1, \ldots, |\Gamma|$ labels the nodes of the quiver diagram and $X^i_{M,M+w_i}$ are the chiral multiplets. [1] As explained in [16], the solutions to these constraints are parameterized by an affine toric variety $\mathcal{Z} \subset \mathbb{C}^{d|\Gamma|}$, which has been called *the variety of commuting matrices* in [1].

The idea which allows describing this as a toric variety can be illustrated with the variety X defined by the simple relation $xy = wz$. Let us solve for z as $z = xy/w$. We can then describe the space of functions on the variety, as the functions $f(w, x, y)$ which are generated by multiplying the monomials w, x and y and the monomial xy/w. In other words, the presence of z is described by admitting more functions than we would on $\mathbb{C}^3$. The set of exponents of these monomials is the cone M_+ generated by positive integral combinations of the vectors $(1\ 0\ 0)$, $(0\ 1\ 0)$, $(0\ 0\ 1)$ and $(-1\ 1\ 1)$.

The same data can be described by giving the dual cone N_+ of vectors satisfying $n \cdot m \geq 0$. In this case it would be generated by $n_a = (1\ 1\ 0)$, $n_b = (1\ 0\ 1)$, $n_c = (0\ 1\ 0)$ and $n_d = (0\ 0\ 1)$.

Now we can describe the space of functions on X by associating variables with these generators of N_+, say a, b, c and d, and writing monomials in these variables. The non-trivial data about X is now expressed in the relations between the generators. In our example there is a single relation, $n_a + n_d = n_b + n_c$.

[1] We are only considering the special case $Y = 0$ here, as this is what makes direct contact with [23] and the resolution to weighted projective spaces. The moduli Y appear to be connected with deformations of bundles which appear after restriction to the CY [9].

The important fact is now that **constraints are dual to gauge invariances**, where duality is in the linear algebra sense. This is fairly obvious on reflection but can be best seen by using the language of exact sequences. Consider a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Its exactness means that $g \cdot f = 0$.

One interpretation we could make of this is that $g = 0$ expresses a set of constraints on the space B, parameterized by elements of C. The map f would then be an explicit set of solutions to the constraints, parameterized by elements of A.

Another possible interpretation is that we have abelian gauge symmetries acting on the space B, described by the image of the map f and parameterized by elements of A. We could then regard C as the gauge invariant subspace or quotient B/A. This formulation is the best when we can use it, as it avoids the need to make an explicit choice of a gauge slice; if we needed to exhibit a slice in B we would need to choose a partial inverse h of g satisfying $g \cdot h = 1|_C$; h would then give a map from C to the slice.

The point now is that duality (in the linear algebra sense) reverses all the arrows and thus the roles of the maps f and g. This leads to duality between constraints and gauge invariances.

In this example, the dual relation between M_+ and N_+ implies that constraints on the M monomials will lead to gauge invariances for the N monomials. This is formalized by writing the space X as the spectrum of the algebra of monomials. Defining this algebra as $\mathrm{Hom}(M, \mathbb{C})$ allows applying the previous discussion; see for example [8].

Thus, the relation $n_a + n_d = n_b + n_c$ of our example should translate into a gauge invariance, with $U(1)$ acting on the four variables $(a\ b\ c\ d)$ with the charges $(1\ -1\ -1\ 1)$. We can test this claim by writing out the gauge invariant monomials and checking that they satisfy the relation. Indeed, these are ab, ac, db and dc, which can be identified with the original w, x, y, and z satisfying the relation $xy = wz$.

This type of realization, in which the relations between generators of N_+ are interpreted as gauge invariances, is completely general, and gives a method for turning the F-flatness constraints into abelian gauge invariances. Thus we can realize the final moduli space of vacua entirely as a toric variety. The main difference between the original abelian gauge invariances and the newly generated ones is that the former will typically come with FI terms, while the latter will not.